

COMPACT IMBEDDING THEOREMS FOR QUASIBOUNDED DOMAINS

BY
ROBERT A. ADAMS⁽¹⁾

1. **Introduction.** Let G be an unbounded open set in Euclidean n -space E_n . In this paper we investigate (for a large class of such domains) the problem of determining for which values of m, p, j and r the Sobolev space imbedding

$$(1) \quad W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$$

is or is not compact. Provided $j < m$ continuous imbeddings of this type are known to exist for $p \leq r \leq np(n - mp + jp)^{-1}$ if $n > mp - jp$ or for $p \leq r < \infty$ if $n \leq mp - jp$ (the Sobolev Imbedding Theorem, e.g. [5, Lemma 5]). If G were bounded Kondrašov's compactness theorem [9] would yield the complete continuity of these imbeddings except in the extreme case $r = np(n - mp + jp)^{-1}$. Such compactness theorems are useful for studying existence and spectral theory for partial differential operators on G .

In a sequence of recent papers the writer [1]–[4] and C. W. Clark [6], [7], [8] have studied such compactness problems for various unbounded domains. It is clear that the imbedding (1) cannot be compact if G contains infinitely many disjoint congruent balls, for if a fixed C^∞ function has support in one of these balls then the set of its translates with supports in the other balls is bounded in any space $W_0^{m,p}(G)$ but is not precompact in any such space. Thus a necessary condition for the compactness of imbedding (1) is that G should be *quasibounded*, i.e. that $\text{dist}(x, \text{bdry } G) \rightarrow 0$ whenever $|x| \rightarrow \infty, x \in G$. In [1] the writer has shown that if $n > 1$ then quasiboundedness is not sufficient for compactness.

The dimension of the boundary of G is a critical factor in determining whether or not (1) is compact. If G is quasibounded and bounded by smooth “reasonably unbroken” $(n-1)$ -dimensional manifolds then (1) is compact [3, Theorem 1] for any m and p and for the same values of j and r as in the case of bounded G . However if G has discrete (0-dimensional) boundary then [2, Theorem 1] no such imbedding can be compact unless $mp > n$.

Our purpose in this paper is to study the compactness of imbedding (1) for quasibounded domains G whose boundaries are comprised of smooth manifolds

Received by the editors June 6, 1969.

⁽¹⁾ Research sponsored by Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under AFOSR grant AFOSR-68-1531.

Copyright © 1970, American Mathematical Society

of various dimensions. Roughly speaking our results are as follows. If k is the smallest integer for which those boundary manifolds of G having dimension not less than $n-k$ bound a quasibounded domain then no imbedding of type (1) can be compact when $mp < k$. On the other hand, if, in addition, the boundary manifolds are "reasonably unbroken" and if $mp > n + p - np/k$ then (1) is compact for the same values of j and r as in the case of bounded G . Our results thus interpolate between the extreme cases mentioned above. We consider first domains G with flat (planar) boundaries, establishing in §2 a necessary condition for the compactness of (1) for such G , and in §3 a slightly stronger sufficient condition. If $m=1$ these conditions are equivalent for certain domains. In §4 similar results are obtained for nonflatly-bounded domains G .

As usual, in this paper $W_0^{m,p}(G)$ denotes, for $p \geq 1$ and $m=0, 1, 2, \dots$, the Sobolev space obtained by completing with respect to the norm

$$(2) \quad \|u\|_{m,p,G} = \left\{ \sum_{j=0}^m |u|_{j,p,G}^p \right\}^{1/p}$$

the space $C_0^\infty(G)$ of all infinitely differentiable, complex functions having compact support in G where

$$|u|_{j,p,G}^p = \sum_{|\alpha|=j} \int_G |D^\alpha u(x)|^p dx.$$

α denotes an n -tuple of nonnegative integers $(\alpha_1, \dots, \alpha_n)$; $|\alpha| = \alpha_1 + \dots + \alpha_n$; $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$; $D_j = \partial/\partial x_j$. Note that $|u|_{0,p,G} = \|u\|_{0,p,G}$ is the norm of u in $L^p(G)$. $W^{m,p}(G)$ represents the completion with respect to the norm (2) of the space of all infinitely differentiable functions on G for which (2) is finite. Provided the boundary of G satisfies certain mild regularity conditions [5, Lemma 5] the Sobolev Imbedding Theorem referred to above, and also (provided G is bounded) the Kondrašov Compactness Theorem, remain valid for imbeddings of $W^{m,p}(G)$. No compactness theorems of this sort are yet known if G is unbounded.

2. Flatly-bounded domains—noncompact imbeddings. Let H be a k -dimensional plane ($0 \leq k \leq n-1$) in E_n and let a be a point on H . With respect to a new system of rectangular coordinates z in E_n having origin at a and obtained from the usual coordinates by an affine transformation, H has equations $z_1 = z_2 = \dots = z_{n-k} = 0$, or more simply $r=0$ where $r = \sum_{i=1}^{n-k} z_i^2$. The coordinate r , together with $n-k-1$ angle coordinates collectively denoted σ and the coordinates $z' = (z_{n-k+1}, \dots, z_n)$ form a system of cylindrical polar coordinates in E_n with origin at a and cylindrical axis H .

The k -tube $T_\delta(H)$ of radius δ and axis H is the set $\{x \in E_n : \text{dist}(x, H) = r < \delta\}$. By a *tube function* for the tube $T_\delta(H)$ we mean a C^∞ function $\theta: E_n \rightarrow [0, 1]$ whose value at x depends only on $r = \text{dist}(x, H)$ and which vanishes identically near H and is identically unity outside $T_\delta(H)$.

LEMMA 1. Let H be a k -plane in E_n and $a \in H$. Let $1 \leq p < \infty$. If $u(x) = v(r)$ where $r = \text{dist}(x, H)$ and $v \in C^{|\alpha|}((0, \infty))$ then for all $x \notin H$

$$|D^\alpha u(x)|^p \leq \text{const} \sum_{j=1}^{|\alpha|} |v^{(j)}(r)|^p r^{pj - p|\alpha|}$$

where the constant depends only on α , p and k .

Proof. Since $z_i = \sum_{j=1}^n c_{ij}(x_j - a_j)$ and so $\partial/\partial x_i = \sum_{j=1}^n c_{ji} \partial/\partial z_j$ we may assume with no loss of generality that H is a coordinate plane and $z = x$. We show that there exist homogeneous polynomials $P_{\alpha,j}(x)$ of degree $|\alpha|$ (possibly the zero polynomial) such that for $r > 0$

$$(3) \quad D^\alpha u(x) = \sum_{j=1}^{|\alpha|} P_{\alpha,j}(x) v^{(j)}(r) r^{j-2|\alpha|}.$$

Since $|P_{\alpha,j}(x)| \leq \text{const } r^{|\alpha|}$ the conclusion of the lemma for $p=1$ follows at once. The result for general p then follows from the well-known inequality

$$(4) \quad \left| \sum_{j=1}^N A_j \right|^p \leq \text{const} \sum_{j=1}^N |A_j|^p$$

where the constant depends only on p and N .

Note that $D^\alpha u(x) = 0$ unless $\alpha_{n-k+1} = \dots = \alpha_n = 0$. If $1 \leq i \leq n-k$ then $D_i u(x) = v'(r)x_i/r$ which is of the required form. Assume (3) holds for all α with $|\alpha| \leq m$. If $|\beta| = m+1$ then $D^\beta = D_i D^\alpha$ for some i , α where $|\alpha| = m$. Applying the induction hypothesis and the chain rule we obtain

$$\begin{aligned} D^\beta u(x) &= \sum_{j=1}^m \{ D_i P_{\alpha,j}(x) v^{(j)}(r) r^{j-2m} + P_{\alpha,j}(x) v^{(j+1)}(r) x_i r^{j-2m-1} \\ &\quad - (2m-j) P_{\alpha,j}(x) v^{(j)}(r) x_i r^{j-2m-2} \} \\ &= \sum_{j=1}^{m+1} P_{\beta,j}(x) v^{(j)}(r) r^{j-2(m+1)} \end{aligned}$$

where $P_{\beta,j}$ is given by

$$\begin{aligned} P_{\beta,1}(x) &= r^2 D_i P_{\alpha,1}(x) - (2m-1) x_i P_{\alpha,1}(x), \\ P_{\beta,j}(x) &= r^2 D_i P_{\alpha,j}(x) - (2m-j) x_i P_{\alpha,j}(x) + x_i P_{\alpha,j-1}(x) \quad \text{if } 2 \leq j \leq m, \\ P_{\beta,m+1}(x) &= x_i P_{\alpha,m}(x). \end{aligned}$$

Clearly $P_{\alpha,j}(x)$ is a polynomial of the desired type and the proof is complete.

LEMMA 2. Let λ be a positive integer and let $r = s^\lambda$, $s > 0$. If $f \in C^j((0, \infty))$ and $1 \leq p < \infty$ then

$$(5) \quad |(d/dr)^j f(r^{1/\lambda})|^p \leq \text{const} \sum_{i=1}^j \lambda^{-ip} s^{ip-j\lambda p} |f^{(i)}(s)|^p$$

where the constant depends only on j and p .

Proof. Again the case of general p follows from the special case $p=1$ via (4). For $p=1$ (5) is an immediate consequence of the formula

$$(6) \quad (d/dr)^j = \lambda^{-j} \sum_{i=1}^j P_{j-i,j}(\lambda) s^{i-\lambda j} (d/ds)^i$$

where $P_{i,j}$ is a polynomial of degree i depending on j . We prove (6) by induction on j . Note that $d/dr = \lambda^{-1} s^{1-\lambda} d/ds$ which is of the required form. Assuming (6) we have

$$\begin{aligned} (d/dr)^{j+1} &= \lambda^{-j} \sum_{i=1}^j P_{j-i,j}(\lambda) \lambda^{-1} s^{1-\lambda} d/ds [s^{i-\lambda j} (d/ds)^i] \\ &= \lambda^{-(j+1)} \sum_{i=1}^j P_{j-i,j}(\lambda) \{s^{i+1-\lambda-\lambda j} (d/ds)^{i+1} + (i-\lambda j) s^{i-\lambda j-\lambda} (d/ds)^i\} \\ &= \lambda^{-(j+1)} \sum_{i=1}^{j+1} P_{j+1-i,j+1}(\lambda) s^{i-\lambda(j+1)} (d/ds)^i \end{aligned}$$

where the polynomials $P_{i,j+1}$ are given by

$$\begin{aligned} P_{0,j+1}(\lambda) &= P_{0,j}(\lambda), \\ P_{i,j+1}(\lambda) &= P_{i,j}(\lambda) + (j+1-i-\lambda j) P_{i-1,j}(\lambda) \quad \text{for } 1 \leq i \leq j-1, \\ P_{j,j+1}(\lambda) &= (1-\lambda j) P_{j-1,j}(\lambda), \end{aligned}$$

which are of the desired form.

LEMMA 3. Let T be a k -tube in E_n with axis H and radius $\delta \leq 1$. Let $1 \leq p < \infty$ and let λ be a positive integer. Then there exists a tube function θ for T satisfying for $|\alpha| > 0$,

$$|D^\alpha \theta(x)|^p \leq \text{const } \lambda^{-p} s^{p-\lambda p|\alpha|}$$

where $s^\lambda = r = \text{dist}(x, H)$ and the constant depends only on α, n, p and k and not on λ .

Proof. Let $f: [0, \infty) \rightarrow [0, 1]$ be a fixed C^∞ function such that $f(s)=0$ near $s=0$ and $f(s)=1$ for $s^\lambda \geq \delta$. Define θ by $\theta(x) = v(r) = f(s)$. Clearly θ is a tube function for T . By Lemmas 1 and 2 we have

$$\begin{aligned} |D^\alpha \theta(x)|^p &\leq \text{const} \sum_{j=1}^{|\alpha|} r^{pj-p|\alpha|} |v^{(j)}(r)|^p \\ &\leq \text{const} \sum_{j=1}^{|\alpha|} \sum_{i=1}^j \lambda^{-ip} s^{ip-\lambda p|\alpha|} |f^{(i)}(s)|^p \\ &\leq \text{const } \lambda^{-p} s^{p-\lambda p|\alpha|}. \end{aligned}$$

The final inequality follows because whenever $D^\alpha \theta(x) \neq 0$, $s < \delta^{1/\lambda} \leq 1$, and also $|f^{(i)}(s)| \leq \text{const}$ for $1 \leq i \leq |\alpha|$.

In the following lemma we consider several $(n-k)$ -tubes in E_n simultaneously. Hence all the related quantities $\theta, r, \sigma, z', s, H$ carry subscripts ranging from 1 to N .

LEMMA 4. Let S be a bounded open set in E_n . Let H_1, \dots, H_N be a finite collection of $(n-k)$ -planes which intersect \bar{S} . Let m be a positive integer and let $\varepsilon > 0$. If either $p > 1$ and $mp \leq k$ or $p = 1$ and $m < k$ then there exists a function $\psi \in C^\infty(E_n)$ with the properties:

- (i) $\psi(x) = 0$ for x near $\bigcup_{i=1}^N H_i$,
- (ii) $0 \leq \psi(x) \leq 1$ for all x ,
- (iii) $\psi(x) = 1$ for x in $E_n - \bigcup_{i=1}^N T_\delta(H_i)$, $\delta > 0$,
- (iv) $\|D^\alpha \psi\|_{0,p,S} \leq \varepsilon$ for $0 < |\alpha| \leq m$.

Proof. We first consider the case that no two of the planes H_i intersect in \bar{S} . It is then possible to choose $\delta \leq 1$ small enough so that if $T_i = T_\delta(H_i)$ then $T_i \cap T_j \cap S$ is empty if $i \neq j$. By Lemma 3 there exist tube functions θ_i for T_i satisfying

$$|D^\alpha \theta_i(x)|^p \leq \text{const } \lambda^{-p} s_i^{p - \lambda p |\alpha|}$$

where $s_i^{\lambda} = r_i = \text{dist}(x, H_i)$ and the constant is independent of λ and α for $0 < |\alpha| \leq m$. Let $\psi(x) = \theta_1(x)\theta_2(x) \cdots \theta_N(x)$. Clearly ψ satisfies (i)–(iii). Note that $D^\alpha \psi = 0$ outside $\bigcup_{i=1}^N T_i$ and that $D^\alpha \psi(x) = D^\alpha \theta_i(x)$ in T_i . We have

$$\begin{aligned} \|D^\alpha \psi\|_{0,p,S}^p &= \sum_{i=1}^N \int_{S \cap T_i} |D^\alpha \theta_i(x)|^p dx \\ &\leq \text{const } \lambda^{-p} \sum_{i=1}^N \int_{S \cap T_i} s_i^{p - \lambda p |\alpha|} r_i^{k-1} dr_i d\sigma_i dz'_i \\ &\leq \text{const } \lambda^{1-p} \int_0^1 s_i^{p - \lambda p |\alpha| + \lambda k - 1} ds_i. \end{aligned}$$

The final constant depends on α, p, n, k, N and $\text{diam } S$ but not on λ . If $|\alpha| \leq m$ and $mp \leq k$ then $p - \lambda p |\alpha| + \lambda k > 0$ and so

$$\|D^\alpha \psi\|_{0,p,S}^p \leq \text{const } \lambda^{1-p} (p + \lambda k - \lambda p |\alpha|)^{-1}.$$

The expression on the right can be made arbitrarily small for sufficiently large λ provided either $p > 1$ or $m < k$. This establishes (iv).

The case of intersecting H_i remains to be considered. Again pick $\delta < 1$ small enough so that $T_i \cap T_j \cap S$ is empty whenever $H_i \cap H_j \cap \bar{S}$ is empty. Define θ_i and ψ as above. The general Leibniz formula states

$$(7) \quad D^\alpha \psi(x) = \sum_{\beta_1 + \dots + \beta_N = \alpha} \binom{\alpha}{\beta_1, \dots, \beta_N} D^{\beta_1} \theta_1(x) \cdots D^{\beta_N} \theta_N(x).$$

For estimates of $|D^\alpha \psi(x)|$ we may drop from terms in the Leibniz expression any factor $D^{\beta_i} \theta_i(x)$ for which $\beta_i = 0$ because $|\theta_i(x)| \leq 1$. For simplicity consider a term $D^{\beta_1} \theta_1(x) \cdots D^{\beta_N} \theta_N(x)$ where no β_i is zero. Decompose S into the union of N subregions S_j such that in S_j we have $s_j \leq s_i$ for $i \neq j$. We now obtain via Lemma 3 in the manner above, noting that whenever $D^{\beta_i} \theta_i(x) \neq 0$ then $s_i < \delta \leq 1$

$$\begin{aligned}
\int_S |D^{\beta_1} \theta_1(x) \cdots D^{\beta_N} \theta_N(x)|^p dx &= \sum_{j=1}^N \int_{S_j} |D^{\beta_1} \theta_1(x) \cdots D^{\beta_N} \theta_N(x)|^p dx \\
&\leq \text{const } \lambda^{-Np} \sum_{j=1}^N \int_{S_j} s_1^{p-\lambda p|\beta_1|} \cdots s_N^{p-\lambda p|\beta_N|} dx \\
&\leq \text{const } \lambda^{-Np} \sum_{j=1}^N \int_{S_j} s_j^{Np-\lambda p|\alpha|} dx \\
&\leq \text{const } \lambda^{1-Np} \int_0^1 s^{Np-\lambda p|\alpha|+\lambda k-1} ds \\
&\leq \text{const } \lambda^{1-Np} (Np+\lambda k-\lambda p|\alpha|)^{-1}.
\end{aligned}$$

Similar estimates can be found for all terms in (7) and (iv) follows once more by taking λ sufficiently large.

DEFINITION. Let G be an open set in E_n . G is called a *regular domain* if $\text{bdry } G = \bigcup_{k=0}^{n-1} G_k$ where G_k is the union of a locally finite collection of smooth manifolds of dimension k in E_n . A regular domain G whose boundary manifolds are all segments of planes of various dimensions will be called a *regular flatly-bounded domain*. An unbounded regular domain G is called *0-quasibounded* if it is quasibounded, i.e. if there exist at most finitely many disjoint congruent balls Q in G , having any specified positive radius, which do not intersect the boundary of G . G is called *k -quasibounded* ($1 \leq k \leq n-1$) if there exist at most finitely many disjoint congruent balls Q in \bar{G} , having any specified positive radius, such that $Q \cap \text{bdry } G \subset \bigcup_{i=0}^{k-1} G_i$; i.e. if $\text{dist}(x, \bigcup_{i=k}^{n-1} G_i) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in G$. For $1 \leq k \leq n-1$ the condition of k -quasiboundedness is stronger than that of $(k-1)$ -quasiboundedness.

THEOREM 1. Let G be a regular, quasibounded, flatly-bounded domain in E_n . Let k be the smallest integer ($1 \leq k \leq n$) for which G is $(n-k)$ -quasibounded. If either $mp \leq k$ and $p > 1$ or $m < k$ and $p = 1$ then no imbedding of the form

$$W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$$

can be compact.

Proof. For $2 \leq k \leq n$ since G is not $(n-k+1)$ -quasibounded there is a sequence of congruent open balls $\{Q_i\}_{i=1}^\infty$ in \bar{G} such that $Q_i \cap \text{bdry } G$ is contained in the union of finitely many $(n-k)$ -planes. If $k=1$ balls Q_i with this property exist trivially since G is regular and flatly-bounded. Let Q denote any one of these balls and let H_1, \dots, H_N be the corresponding $(n-k)$ -planes. Let $\varphi \in C_0^\infty(Q)$ be a function for which

$$\|\varphi\|_{0,r,Q} = 2C > 0, \quad \|\varphi\|_{m,p,Q} = K < \infty.$$

There exists a constant M such that for all $x \in E_n$ and for all α with $0 \leq |\alpha| \leq m$, $|D^\alpha \varphi(x)| \leq M$. Choose $\delta_0 > 0$ small enough so that the sum of the volumes of the intersections with Q of the $(n-k)$ -tubes $T_{\delta_0}(H_i)$, $i=1, \dots, N$, does not exceed $(C/M)^r$. Let

$$\varepsilon = K \left[M \sum_{|\alpha| \leq m} \sum_{\beta < \alpha} \binom{\alpha}{\beta} \right]^{-1}.$$

By Lemma 4 there exists for some $\delta \leq \delta_0$ a function $\psi \in C_0^\infty(E_n - \bigcup_{i=1}^N H_i)$ satisfying $0 \leq \psi(x) \leq 1$, $\psi(x) = 1$ outside $\bigcup_{i=1}^N T_\delta(H_i)$ and $\|D^\alpha \psi\|_{0,p,Q} \leq \varepsilon$ for $0 < |\alpha| \leq m$. Let $\gamma = \varphi \cdot \psi = \varphi - \varphi(1 - \psi)$. Clearly $\gamma \in C_0^\infty(Q \cap G)$. Putting $T_i = T_\delta(H_i)$ we have

$$\begin{aligned} \|\gamma\|_{0,r,G} &\geq \|\varphi\|_{0,r,Q} - \|\varphi\|_{0,r,Q \cap (T_1 \cup \dots \cup T_N)} \\ &\geq 2C - M[\text{vol } Q \cap (T_1 \cup \dots \cup T_N)]^{1/r} \geq C. \end{aligned}$$

Moreover, for $0 < |\alpha| \leq m$

$$\begin{aligned} D^\alpha \gamma &= \psi D^\alpha \varphi + \sum_{\beta < \alpha} \binom{\alpha}{\beta} D^\beta \varphi D^{\alpha-\beta} \psi, \\ \|D^\alpha \gamma\|_{0,p,Q} &\leq \|D^\alpha \varphi\|_{0,p,Q} + \sum_{\beta < \alpha} \binom{\alpha}{\beta} M\varepsilon \leq 2K. \end{aligned}$$

Thus if $K_1^p = \sum_{|\alpha| \leq m} 1$ we have $\|\gamma\|_{m,p,G} \leq 2KK_1$.

Now let φ_i , $i = 1, 2, \dots$ be a translate of φ with support in Q_i and let γ_i be constructed from φ_i as γ from φ above, so that

$$\|\gamma_i\|_{0,r,G} \geq C, \quad \|\gamma_i\|_{m,p,G} \leq 2KK_1.$$

The sequence $\{\gamma_i\}$, though bounded in $W_0^{m,p}(G)$ has no subsequence converging in $L'(G)$. In fact if $i \neq j$ then $\|\gamma_i - \gamma_j\|_{0,r,G} \geq 2^{1/r}C$. Thus the imbedding

$$W_0^{m,p}(G) \rightarrow W_0^{0,r}(G) = L'(G)$$

if it exists cannot be compact. Neither can the imbedding $W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$ for if this latter imbedding were compact then so would be the composition

$$W_0^{m,p}(G) \rightarrow W_0^{j,r}(G) \rightarrow L'(G).$$

3. Flatly-bounded domains—compact imbeddings. Let H be a k -plane in E_n and let $a \in H$. We denote by $T_{\delta,\rho}(H, a)$ the *tube segment* of radius δ and length 2ρ having axis H and centre a . Thus, if P is the orthogonal projection operator on H then

$$T_{\delta,\rho}(H, a) = \{x \in E_n : \text{dist}(x, H) < \delta, \text{dist}(Px, a) < \rho\}.$$

LEMMA 5. *Let H be an $(n-k)$ -plane in E_n ($1 \leq k \leq n$) and let $a \in H$. If either $p > k$ or $p = k = 1$ then there exists a constant depending only on p and k such that for each $\delta, \rho > 0$*

$$\|\gamma\|_{0,p,T} \leq \text{const } \delta |\gamma|_{1,p,T}$$

for all $\gamma \in C_0^\infty(E_n - H \cap T)$ where $T = T_{\delta,\rho}(H, a)$.

Proof. First consider the case $p > k$. Let (r, σ, z') denote cylindrical polar coordinates in E_n with origin at a and cylindrical axis H . By Hölder's inequality for $(r, \sigma, z') \in T$

$$\begin{aligned}
|\gamma(r, \sigma, z')|^{p r^{k-1}} &\leq \delta^{k-1} \left| \int_0^r \frac{d}{dt} \gamma(t, \sigma, z') dt \right|^p \\
&\leq \delta^{k-1} \int_0^\delta |D_t \gamma(t, \sigma, z')|^{p t^{k-1}} dt \left\{ \int_0^\delta t^{-(k-1)/(p-1)} dt \right\}^{p-1} \\
&\leq \left(\frac{p-1}{p-k} \right)^{p-1} \delta^{p-1} \int_0^\delta |D_t \gamma(t, \sigma, z')|^{p t^{k-1}} dt.
\end{aligned}$$

Integrating over Σ and Z the domains of the variables σ and z' respectively in T we obtain

$$\begin{aligned}
\|\gamma\|_{0,p,T}^p &= \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |\gamma(r, \sigma, z')|^{p r^{k-1}} dr \\
&\leq \left(\frac{p-1}{p-k} \right)^{p-1} \delta^{p-1} \int_0^\delta dr \int_\Sigma d\sigma \int_Z dz' \int_0^\delta |D_t \gamma(t, \sigma, z')|^{p t^{k-1}} dt \\
&\leq \text{const } \delta^p |\gamma|_{1,p,T}^p.
\end{aligned}$$

For the special case $p=k=1$ we have

$$\begin{aligned}
|\gamma(x)| &= |\gamma(r, z')| \leq \int_0^\delta |D_t \gamma(t, z')| dt, \\
\|\gamma\|_{0,1,T} &\leq \int_Z dz' \int_0^\delta dr \int_0^\delta |D_t \gamma(t, z')| dt \leq \delta |\gamma|_{1,1,T}.
\end{aligned}$$

COROLLARY. *Under the conditions of the lemma, if $1 \leq q \leq p$ then there exists a constant depending only on p, q, k, n such that for all $\delta > 0$*

$$\|\gamma\|_{0,q,T} \leq \text{const } \delta^{1+n/q-n/p} |\gamma|_{1,p,T}$$

for all $\gamma \in C_0^\infty(E_n - H \cap T)$ where $T = T_{\delta,\delta}(H, a)$.

Proof. By Hölder's inequality and since $\text{vol } T = \text{const } \delta^n$

$$\begin{aligned}
\|\gamma\|_{0,q,T} &\leq \|\gamma\|_{0,p,T} [\text{vol } T]^{1/q-1/p} \\
&\leq \text{const } \delta^{1+n/q-n/p} |\gamma|_{1,p,T}.
\end{aligned}$$

DEFINITION. Let G be an unbounded, regular, flatly-bounded domain in E_n . We shall say that G has the k -tube property if for every sufficiently large positive number R there exists a positive number $\delta = \delta(R)$ with the properties:

(i) $\delta(R) \rightarrow 0$ as $R \rightarrow \infty$,

(ii) for each $x \in G_R = \{y \in G : |y| > R\}$ there exists a k -plane H and a point $a \in H$ such that $x \in T_{\delta,\delta}(H, a)$ and $H \cap T_{2\delta,2\delta}(H, a) \subset \text{bdry } G$.

It is clear that if G has the k -tube property then G is k -quasibounded. Of course the converse is not true as the planar segments comprising the boundary of G may have too many gaps to satisfy condition (ii). For domains G whose boundaries consist only of whole planes the k -tube property is equivalent to k -quasibounded-

ness. Other examples of domains with the k -tube property are not difficult to construct—for example Clark's "spiny urchin" [7] has the 1-tube property in E_2 .

LEMMA 6 (A VARIANT ON POINCARÉ'S INEQUALITY). *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$ -tube property for some k ($1 \leq k \leq n$). If $1 \leq r \leq p$ where either $p > k$ or $p = k = 1$ then there exists a constant depending only on n, k, p and r such that for all $u \in W_0^{1,p}(G)$ and all sufficiently large R*

$$\|u\|_{0,r,G_R} \leq \text{const} [\delta(R)]^{1+n/r-n/p} |u|_{1,p,G}.$$

Proof. Fix R large enough so that $\delta = \delta(R)$ exists. If α is an n -tuple of integers (not necessarily nonnegative) let $Q_\alpha = \{x \in E_n : \alpha_i n^{-1/2} \delta \leq x_i \leq (\alpha_i + 1)n^{-1/2} \delta\}$. Then $E_n = \bigcup_\alpha Q_\alpha$. If $x \in G_R$ then $x \in Q_\alpha$ for some α and there exists an $(n-k)$ -plane H and a point $a \in H$ such that $x \in T_{\delta,\delta}(H, a) = T$ and $H \cap T' \subset \text{bdry } G$ where $T' = T_{2\delta,2\delta}(H, a)$. Clearly $Q_\alpha \subset T'$. For any $\gamma \in C_0^\infty(G)$ since γ vanishes near $H \cap T'$ we have by the corollary of Lemma 5

$$\begin{aligned} \|\gamma\|_{0,r,Q_\alpha \cap G_R} &\leq \|\gamma\|_{0,r,T'} \\ &\leq \text{const} (2\delta)^{1+n/r-n/p} |\gamma|_{1,p,T'} \\ &\leq \text{const} \delta^{1+n/r-n/p} |\gamma|_{1,p,Q'_\alpha} \end{aligned}$$

where Q'_α is the union of all the cubes Q_β which intersect T' . There is a number N depending only on n such that any $N+1$ of the sets Q'_α have empty intersection. Summing the above inequality over all α for which Q_α meets G_R we obtain

$$\|\gamma\|_{0,r,G_R} \leq \text{const } N \cdot \delta^{1+n/r-n/p} |\gamma|_{1,p,G}$$

and this inequality extends by completion from $C_0^\infty(G)$ to $W_0^{1,p}(G)$.

THEOREM 2. *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$ -tube property ($1 \leq k \leq n$). If either $p > k$ or $p = k = 1$ then the imbedding $W_0^{m+1,p}(G) \rightarrow W_0^{m,r}(G)$ (exists and) is compact for $m=0, 1, 2, \dots$ and $1 \leq r < \infty$ if $p \geq n$ or for $1 \leq r < np(n-p)^{-1}$ if $p < n$.*

Proof. First consider the case $1 \leq r \leq p$, $m=0$. To prove that the imbedding $W_0^{1,p}(G) \rightarrow L^r(G)$ is compact we use the following compactness criterion for sets in $L^r(G)$: a sequence $\{u_i\}_{i=1}^\infty$ which is bounded in $L^r(G)$ is precompact in $L^r(G)$ provided

- (a) for every bounded $G' \subset G$ the sequence $\{u_i|_{G'}\}$ is precompact in $L^r(G')$, and
- (b) for each $\varepsilon > 0$ there exists $R > 0$ such that for all i , $\|u_i\|_{0,r,G_R} < \varepsilon$.

Lemma 6 and condition (i) of the $(n-k)$ -tube property assures us that (b) is satisfied for any sequence $\{u_i\}$ bounded in $W_0^{1,p}(G)$. To establish (a) let G' be a bounded subset of G . Then for some R , $G' \subset K_R = \{x \in E_n : |x| \leq R\}$. Let $W^{1,p}(G, R)$ denote the completion with respect to the norm $\|\cdot\|_{1,p,G \cap K_R}$ of the space $C_0^\infty(G)$. The imbedding $W^{1,p}(K_R) \rightarrow L^r(K_R)$ is known to be compact (Kondrašov's Theorem) and since an element of $W^{1,p}(G, R)$ can be extended to be zero outside its support so as to belong to $W^{1,p}(K_R)$ it follows that $W^{1,p}(G, R)$ is compactly imbedded in

$L'(G \cap K_R)$. But $\{u_i|K_R\}$ is bounded in $W^{1,p}(G, R)$ and hence precompact in $L'(G \cap K_R)$ whence $\{u_i|G'\}$ is precompact in $L'(G')$ as required.

By Sobolev's Imbedding Theorem $W_0^{1,p}(G)$ is continuously imbedded in $L^q(G)$ for any q satisfying $p \leq q < \infty$ if $p \geq n$ or $p \leq q \leq np(n-p)^{-1}$ if $p < n$. Select such a q and a sequence $\{u_i\}$ bounded in $W_0^{1,p}(G)$ so that, say, $\|u_i\|_{0,q,G} \leq C$. We may assume, passing to a subsequence if necessary, that $\{u_i\}$ converges in $L^p(G)$. By Hölder's Inequality if $p \leq r < q$

$$\begin{aligned} \|u_i - u_j\|_{0,r,G} &\leq \|u_i - u_j\|_{0,p,G}^\lambda \|u_i - u_j\|_{0,q,G}^{1-\lambda} \\ &\leq (2C)^{1-\lambda} \|u_i - u_j\|_{0,p,G}^\lambda \end{aligned}$$

where $\lambda = p(q-r)r^{-1}(q-p)^{-1} > 0$. Hence $\{u_i\}$ converges in $L^r(G)$ and so the imbedding $W_0^{1,p}(G) \rightarrow L^r(G)$ is compact for $1 \leq r < \infty$ if $p \geq n$ and for $1 \leq r < np(n-p)^{-1}$ if $p < n$.

Finally, if $\{u_i\}$ is bounded in $W_0^{m+1,p}(G)$ then for any α with $0 \leq |\alpha| \leq m$, $\{D^\alpha u_i\}$ is bounded in $W_0^{1,p}(G)$ and so has a subsequence convergent to an element v_α of $L^r(G)$. In particular (for a suitable subsequence) $u_i \rightarrow v_0$ in $L^r(G)$ and so in the sense of distributions. Since $D^\alpha u_i \rightarrow v_\alpha$ in $L^r(G)$ and $D^\alpha u_i \rightarrow D^\alpha v_0$ in the sense of distributions it follows that $v_\alpha = D^\alpha v_0$ and $u_i \rightarrow v_0$ in $W_0^{m,r}(G)$. This completes the proof.

This theorem affords for imbeddings of the sort $W_0^{1,p}(G) \rightarrow L^r(G)$ on domains G for which $(n-k)$ -quasiboundedness is equivalent to the $(n-k)$ -tube property, a complete converse to Theorem 1. For imbeddings $W_0^{m,p}(G) \rightarrow L^r(G)$, $m \geq 2$, we do not fare quite so well.

THEOREM 3. *Let G be an unbounded, regular, flatly-bounded domain in E_n having the $(n-k)$ -tube property ($1 \leq k \leq n$). Then the imbedding*

$$W_0^{m,p}(G) \rightarrow W_0^{j,r}(G), \quad 0 \leq j < m,$$

is compact in any of the following cases:

- (i) $m=p=k=1$,
- (ii) $mp > n+p-np/k$, $p \leq r < p^*$,
- (iii) $mp > n+(j+1)p-np/k$, $1 \leq r < p^*$,

where $p^ = np(n-mp+jp)^{-1}$ if $n > mp-jp$ and $p^* = \infty$ if $n \leq mp-jp$.*

Proof. The case $m=1$ has already been proved. If $m \geq 2$ the imbedding $W_0^{m,p}(G) \rightarrow W_0^{1,q}(G)$ is continuous for $p \leq q \leq np(n-mp+p)^{-1}$ if $n > mp-p$ and $p \leq r < \infty$ if $n \leq mp-p$. By Theorem 2 the imbedding $W_0^{1,q}(G) \rightarrow L^p(G)$ is compact provided $q > k$. Since $mp > n+p-np/k$ is equivalent to $np(n-mp+p)^{-1} > k$ such $q > k$ can always be chosen and so the composed imbedding $W_0^{m,p}(G) \rightarrow L^p(G)$ is compact.

By a standard interpolation theorem for Sobolev spaces [5, Lemma 6] there exists a constant K such that for $0 \leq j < m$, $p \leq q < p^*$ we have

$$\|u\|_{j,r,G} \leq K \|u\|_{m,p,G}^\lambda \|u\|_{0,p,G}^{1-\lambda}$$

for all $u \in W_0^{m,p}(G)$ where $\lambda = (nr + jrp - np)(mrp)^{-1}$. Note that $0 \leq \lambda < 1$ for all relevant values of j, m, n, r and p . If $\{u_i\}_{i=1}^\infty$ is a bounded sequence in $W_0^{m,p}(G)$ then it has a subsequence again denoted $\{u_i\}$ which is convergent in $L^p(G)$. Since

$$\begin{aligned} \|u_i - u_k\|_{j,r,G} &\leq K \|u_i - u_k\|_{m,p,G}^\lambda \|u_i - u_k\|_{0,p,G}^{1-\lambda} \\ &\leq K [2 \sup \|u_i\|_{m,p,G}]^\lambda \|u_i - u_k\|_{0,p,G}^{1-\lambda}, \end{aligned}$$

it follows that $\{u_i\}$ is a Cauchy sequence and hence convergent in $W_0^{j,r}(G)$ proving case (ii).

If $mp > n + (j+1)p - np/k$ (and in particular if $j=0$ in case (ii)) we have by Sobolev's theorem and Theorem 2

$$W_0^{m,p}(G) \rightarrow W_0^{j+1, np(n-mp+jp+p)^{-1}}(G) \rightarrow W_0^{j,r}(G)$$

the second imbedding being compact since $np(n-mp+jp+p)^{-1} > k$.

REMARK. The condition $mp > n + p - np/k$ implies $mp > k$. The converse is, however, true for $m \geq 2$ only if $k = n$ or $k \leq p$. Thus even for domains G for which $(n-k)$ -quasiboundedness is equivalent to the $(n-k)$ -tube property imbeddings of $W_0^{m,p}(G)$ corresponding to the cases $1 < p < k < mp \leq n + p - np/k$ and $1 = p < k \leq m \leq n + 1 - n/k$ fail to be covered either by Theorem 1 or Theorem 3.

4. Extensions to nonflatly-bounded domains. Let G, G' be open sets in E_n . A one-to-one transformation M from G onto G' is called an m -diffeomorphism of modulus C if all the components of M and M^{-1} have continuous partial derivatives of all orders up to and including m , and these partials do not exceed C in modulus.

LEMMA 7. Let G, G' be open in E_n and let M be an m -diffeomorphism of modulus C from G onto G' . Let

$$Au(y) = u(M^{-1}y), \quad y \in G'.$$

Then A is a homeomorphism from $W^{m,p}(G)$ [respectively $W_0^{m,p}(G)$] onto $W^{m,p}(G')$ [resp. $W_0^{m,p}(G')$] and there exist constants C_1 and C_2 depending only on n, p and C and not on G or G' such that for all $u \in W^{m,p}(G)$

$$C_1 \|u\|_{m,p,G} \leq \|Au\|_{m,p,G'} \leq C_2 \|u\|_{m,p,G}.$$

Proof. A is a homeomorphism from $L^p(G)$ onto $L^p(G')$ for if $\partial M / \partial x$ represents the Jacobian determinant of M then

$$\begin{aligned} \left\{ \sup_{y \in G'} \left| \frac{\partial M^{-1}(y)}{\partial y} \right|^{1/p} \right\}^{-1} \|u\|_{0,p,G} &\leq \|Au\|_{0,p,G'} \\ &\leq \sup_{x \in G} \left| \frac{\partial M(x)}{\partial x} \right|^{1/p} \|u\|_{0,p,G}. \end{aligned}$$

By induction and formal applications of the chain rule it is easily verified that in the sense of distributions on G'

$$D^\alpha(Au) = \sum_{|\beta| \leq |\alpha|} M_{\alpha\beta} A(D^\beta u)$$

where $M_{\alpha\beta}$ is a polynomial of degree $|\beta|$ in the derivatives of the components of M^{-1} involving derivatives of orders not exceeding $|\alpha|$. It follows that

$$\begin{aligned}\|Au\|_{m,p,C'}^p &= \sum_{|\alpha| \leq m} \left\| \sum_{|\beta| \leq |\alpha|} M_{\alpha\beta} A(D^\beta u) \right\|_{0,p,G'}^p \\ &\leq \text{const} \sum_{|\beta| \leq m} \|A(D^\beta u)\|_{0,p,G'}^p \\ &\leq \text{const} \|u\|_{m,p,G}^p.\end{aligned}$$

The reverse inequality follows in a similar manner.

LEMMA 8. *Under the hypotheses of Lemma 7 $W^{m,p}(G)$ is compactly imbedded in $W^{j,r}(G)$ if and only if $W^{m,p}(G')$ is compactly imbedded in $W^{j,r}(G')$. A similar statement holds for the spaces $W_0^{m,p}(G)$.*

Proof. Suppose the imbedding $W^{m,p}(G) \rightarrow W^{j,r}(G)$ is compact. Let $\{u_i\}_{i=1}^\infty$ be a bounded sequence in $W^{m,p}(G')$. Then $\{A^{-1}u_i\}$ is bounded in $W^{m,p}(G)$ and so has a subsequence converging in $W^{j,r}(G)$. The corresponding subsequence of $\{u_i\}$ is convergent in $W^{j,r}(G')$ whence the imbedding $W^{m,p}(G') \rightarrow W^{j,r}(G')$ is compact. The other cases are proved similarly.

Of course Lemma 8 can be used to obtain immediately the conclusions of Theorems 1–3 for any domain G which is m -diffeomorphic to an unbounded, regular, flatly-bounded domain G' satisfying the conditions of the particular theorem. As most quasibounded domains do not have this property we obtain generalizations of these theorems with localized hypotheses.

THEOREM 4. *Let G be a regular, unbounded domain in E_n . Let k be the largest integer ($1 \leq k \leq n$) for which for some constant C there exist infinitely many mutually disjoint open sets U in \bar{G} each of which is m -diffeomorphic with modulus not greater than C to the unit ball B in E_n in such a way that $U \cap \text{bdry } G$ is mapped into a subset of the union of finitely many $(n-k)$ -planes. (In particular G is not $(n-k+1)$ -quasi-bounded.) If either $mp \leq k$, $p > 1$ or $m < k$, $p = 1$ then no imbedding of the form $W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$ can be compact.*

Proof. Let $\{U_i\}_{i=1}^\infty$ be a sequence of mutually disjoint open sets in \bar{G} for which there correspond m -diffeomorphisms $M_i: U_i \rightarrow B$ having modulus $\leq C$ and such that $M_i(U_i \cap \text{bdry } G) \subset B \cap P_i$ where P_i is the union of finitely many $(n-k)$ -planes. Let $\varphi \in C_0^\infty(B)$ be such that

$$\|\varphi\|_{0,r,B} = C_1 > 0, \quad \|\varphi\|_{m,p,B} = K_1 < \infty.$$

By the method used in the proof of Theorem 1 we can construct functions $\gamma_i \in C_0^\infty(B - P_i)$ such that

$$\|\gamma_i\|_{0,r,B} \geq C_2 > 0, \quad \|\gamma_i\|_{m,p,B} \leq K_2 < \infty,$$

the constants C_2 and K_2 being independent of i . Denoting by A_i the operator for

which $A_i u(y) = u(M_i^{-1}y)$, $y \in B$ we have by Lemma 7 that there exist constants C_3 and K_3 again independent of i such that

$$\|A_i^{-1}\gamma_i\|_{0,r,G} \geq C_3 > 0, \quad \|A_i^{-1}\gamma_i\|_{m,p,G} \leq K_3 < \infty$$

and

$$A_i^{-1}\gamma_i \in C_0^\infty(U_i \cap G).$$

The noncompactness of the imbedding $W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$ now follows as in Theorem 1.

As an analog of the compactness Theorems 2 and 3 we have

THEOREM 5. *Let G be an unbounded open set in E_n with the property that there exist constants C , R_0 and K such that for each $R \geq R_0$ there exist positive numbers $d(R)$ and $\delta(R)$ with the following properties:*

- (i) $d(R) + \delta(R) \rightarrow 0$ as $R \rightarrow \infty$,
- (ii) $d(R)/\delta(R) \leq K$, $R \geq R_0$,
- (iii) *for each $x \in G_R = \{x \in G : |x| > R\}$ the ball $B_\delta(x)$ of radius $\delta(R)$ and center x can be mapped by a 1-diffeomorphism M of modulus $\leq C$ onto a set S in E_n such that for some $(n-k)$ -plane H ($1 \leq k \leq n$) and some point $a \in H$ we have $S \subset T_{d(R),d(R)}(H, a)$ and $H \cap T_{d(R),d(R)}(H, a) \subset M(\text{bdry } G \cap B_\delta(x))$.*

Then the imbedding $W_0^{m,p}(G) \rightarrow W_0^{j,r}(G)$, $0 \leq j < m$ is compact in any of the following cases:

- (a) $m = p = k = 1$,
 - (b) $mp > n + p - np/k$, $p \leq r < p^*$,
 - (c) $mp > n + (j+1)p - np/k$, $1 \leq r < p^*$,
- where $p^* = np(n - mp + jp)^{-1}$ if $n > mp - jp$ and $p^* = \infty$ if $n \leq mp - jp$.

Proof. The conclusion is the same as that of Theorem 3 and the proof is identical if we reprove Lemma 6 (Poincaré's inequality) under the conditions of this theorem. Thus, let $p > k$ or $p = k = 1$ and let $1 \leq r \leq p$. Fix $R \geq R_0$ and let $d = d(R)$ and $\delta = \delta(R)$. Define the cubes Q_α as in the proof of Lemma 6. If $x \in G_R$ then for some α , $x \in Q_\alpha \subset B_\delta(x)$. There exists a 1-diffeomorphism M of $B = B_\delta(x)$ onto $S \subset E_n$ having modulus $\leq C$ and there exists an $(n-k)$ -plane H and a point $a \in H$ such that $S \subset T = T_{d,d}(H, a)$ and $H \cap T \subset M(B \cap \text{bdry } G)$. For any $\gamma \in C_0^\infty(G)$ we have that $A\gamma$ (defined by $A\gamma(y) = \gamma(M^{-1}y)$, $y \in S$) vanishes near $H \cap T$. Thus by the corollary of Lemma 5, Lemma 7 and the fact that $d \leq K\delta$

$$\begin{aligned} \|\gamma\|_{0,r,Q_\alpha \cap G_R} &\leq \|\gamma\|_{0,r,B} \leq \text{const} \|A\gamma\|_{0,r,S} \\ &\leq \text{const} d^{1+n/r-n/p} \|A\gamma\|_{1,p,T} \\ &\leq \text{const} \delta^{1+n/r-n/p} \|\gamma\|_{1,p,M^{-1}(T)} \\ &\leq \text{const} \delta^{1+n/r-n/p} \|\gamma\|_{1,p,Q'_\alpha} \end{aligned}$$

where Q'_α is the union of all the cubes Q_β which intersect $M^{-1}(T)$. Since the modulus of M is bounded, M^{-1} is Lipschitzian and there exists a constant λ such that $M^{-1}(T) \subset B_{\lambda d}(x) \subset B_{\lambda K \delta}(x)$. Thus there is a constant N independent of R and x

such that any $N+1$ of the sets Q'_α have empty intersection. Summing the above inequality over those α for which Q_α meets G_R we obtain, as in Lemma 6, the required form of Poincaré's inequality.

5. An application to differential operators. Let L be a linear partial differential operator of order $2m$ in G given by

$$Lu(x) = \sum_{|\alpha| \leq 2m} a_\alpha(x) D^\alpha u(x)$$

with coefficients a_α infinitely differentiable, bounded, complex functions on G . Suppose L is such that it satisfies the boundedness condition

$$\left| \int_G L\varphi(x) \overline{\psi(x)} dx \right| \leq c_0 \|\varphi\|_{m,2,G} \|\psi\|_{m,2,G}$$

for all $\varphi, \psi \in C_0^\infty(G)$, and also Garding's inequality

$$\operatorname{Re} \int_G L\varphi(x) \overline{\varphi(x)} dx \geq c_1 \|\varphi\|_{m,2,G}^2 - c_2 \|\varphi\|_{0,2,G}^2$$

for all $\varphi \in C_0^\infty(G)$, where $c_0, c_1 > 0$ and c_2 are constants. The realization of L in $L^2(G)$ corresponding to null Dirichlet boundary data is an operator T in $L^2(G)$ defined by

$$\begin{aligned} \operatorname{Dom}(T) &= W_0^{m,2}(G) \cap \{f \in L^2(G) : Lf \in L^2(G)\} \\ Tf &= Lf, \quad f \in \operatorname{Dom}(T). \end{aligned}$$

THEOREM 6. *If G is open in E_n and satisfies the conditions of either Theorem 3 or Theorem 5 with $2m > n + 2 - 2n/k$ then T as defined above is a closed linear operator in $L^2(G)$; the spectrum $\sigma(T)$ is discrete and has no finite limit points; for $\lambda \notin \sigma(T)$ the resolvent operator $R_\lambda(T) = (\lambda I - T)^{-1}$ is completely continuous.*

The proof is identical to that of the standard theorem of this type. A sketch can be found in [6].

REFERENCES

1. R. A. Adams, *Compact Sobolev imbeddings for unbounded domains*, Pacific J. Math. (to appear).
2. ———, *Compact Sobolev imbeddings for unbounded domains with discrete boundaries*, J. Math. Anal. Appl. **24** (1968), 390–394.
3. ———, *The Rellich-Kondrachov theorem for unbounded domains*, Arch. Rational Mech. Anal. **29** (1968), 370–394. MR **37** #3349.
4. ———, *Compact Sobolev imbeddings for "pepper sets"*, J. Math. Anal. Appl. **27** (1969), 405–408.
5. F. E. Browder, *On the spectral theory of elliptic differential operators. I*, Math. Ann. **142** (1960/61), 22–130. MR **35** #804.
6. C. W. Clark, *An embedding theorem for function spaces*, Pacific J. Math. **19** (1966), 243–251. MR **34** #4946.

7. C. W. Clark, *Rellich's embedding theorem for a "spiny urchin"*, Canad. Math. Bull. **10** (1967), 731–734. MR 37 #1989.
8. ———, *Some embedding theorems for Sobolev spaces*, (to appear).
9. V. I. Kondrašov, *On certain properties of functions in the space L_p* , Dokl. Akad. Nauk SSSR **48** (1945), 563–566.

UNIVERSITY OF BRITISH COLUMBIA,
VANCOUVER, B.C., CANADA